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THREE-DIMENSIONAL BUCKLING ANALYSIS OF LAMINATED COMPOSITE HOLLOW CYLINDERS AND CYLINDRICAL PANELS

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Abstract—The buckling problem of thick, cross-ply laminated cylinders and cylindrical panels under combined external loading is investigated on the basis of fully three-dimensional elasticity considerations. The three-dimensional pre-buckling state that is initially employed, assumes zero shear stresses and is suitable for the analysis of both open panels and closed cylinders. This assumption leads to a set of three-dimensional linearized buckling equations. Both sets of three-dimensional prebuckling and buckling equations are then solved on the basis of a recursive method. The analysis is suitable for studying the buckling behavior of simply supported cross-ply cylinders, subjected to the single or the combined action of an axial compression and a uniform lateral pressure, or open cross-ply cylindrical panels under axial compression. The numerical results presented and discussed throughout this paper deal with cylinders and cylindrical panels having a symmetric or an antisymmetric cross-ply lay-up.

INTRODUCTION

The history of theoretical buckling analyses of cylindrical shells subjected to either axial compression or lateral pressure is well documented in the literature, by means of several books, monographs and review papers (e.g. Simitses, 1986). Based almost exclusively on classical shell theory investigations, it started in the beginning of this century with the primary interest being focused on failure of shells made of steel, a material that is considered as macroscopically isotropic. Today, the main interest of such investigations is related to composite materials applications.

Although it is well known that composite thin-walled structures are considerably more sensitive in transverse deformation effects than corresponding structures made from isotropic materials, buckling analyses of composite cylindrical shells are still based on two-dimensional shell theories. These, however, are mainly refined, higher-order shell theories that take transverse shear deformation effects into consideration (e.g. Palazoto and Linnemann, 1991; Simitses and Anastasiadis, 1991, 1992; Soldatos, 1992a,b,c). To the authors' best knowledge, the only analytical three-dimensional buckling studies existing in the cylindrical shell literature are due to Babich and Kilin (1985) and Kardomateas (1993a,b). On the basis of axisymmetric but still three-dimensional pre-buckling state considerations, Babich and Kilin (1985) dealt with the buckling of a three-layered orthotropic cylinder under axial compression, while Kardomateas (1993a) studied the plane strain buckling of an infinite hollow orthotropic cylinder subjected to a uniform lateral pressure. Since the linearized buckling equations obtained were differential equations with variable coefficients, their solution was sought, in both papers (Babich and Kilin, 1985; Kardomateas, 1993a), on the basis of numerical methods.

Dealing with buckling of transversely isotropic, complete, simply supported cylindrical shells under axial compression, Kardomateas (1993b) obtained an exact solution of three-dimensional buckling equations in terms of Bessel functions. This was achieved on the basis of an essentially membrane pre-buckling state, in which a constant axial normal stress is the only non-zero initial stress component. In this connection, Noor and Peters' (1989) computational procedure is worth mentioning, since it also employs a membrane pre-buckling state. Using a two-field mixed finite element method for the discretization in the thickness direction, this (Noor and Peters, 1989) was applied for three-dimensional buckling analysis of complete, simply supported, cross-ply laminated cylinders under axial compression.

On the basis of fully three-dimensional elasticity considerations, this paper studies the buckling behaviour of simply supported: (1) hollow laminated cylinders subjected to combined axial compression and uniform lateral pressure, and (2) open laminated cylindrical panels under axial compression. To the authors' best knowledge, the three-dimensional pre-buckling analysis employed for either open or closed laminated cylinders has not been considered in the literature. In more detail, all initial shear stresses are assumed to be zero, while all three initial normal stresses have a non-zero contribution both to the pre-buckling state and to the buckling analysis. In contrast with corresponding two-dimensional analyses, these non-zero initial stresses are all varying through the thickness of the cylindrical shell or panel considered and may arise due to the single or the combined action of an axial compression or a lateral pressure. As a result, the buckling problem considered is not related to the corresponding free vibration problem in the close connection observed when a two-dimensional theory, that assumes a membrane prebuckling state, is employed.

Both three-dimensional sets of prebuckling and linearized buckling equations obtained are solved on the basis of a recursive formulation of a successive approximation approach. For the buckling analysis, this formulation is equivalent to the one employed recently in connection with exact three-dimensional dynamic analyses of homogeneous and laminated composite cylinders and panels (Soldatos and Hadjigeorgiou, 1990; Soldatos, 1991; Soldatos and Hawkes, 1991; Hawkes and Soldatos, 1992). Compared, however, to the previous analyses, this new formulation considerably facilitates the numerical calculations in the sense that it always yields buckling loads as roots of a 6×6 eigen-determinant. For the pre-buckling analysis, this new formulation always ends up with a set of two simultaneous linear algebraic equations whose solution provides with the pre-buckling stress distributions regardless of the number of the material layers of the shell considered.

Dealing with buckling problems that involve cross-ply laminated configurations, a difficulty involved in mathematical modelling is more evident than it is in the corresponding homogeneous case (Soldatos and Ye, 1994). It arises from the fact that, upon applying, for instance, a uniform axial compression acting directly on the curved edges, different layers that possess different material properties undergo different axial contractions. Following, however, the Babich and Kilin (1985) pre-buckling analysis, this difficulty is overcome here by assuming that the axial compression is applied on the curved edges by means of a rigid plane that remains always normal to the axis of the cylinder and causes, therefore, the same initial axial normal strain in all layers. Under these considerations, the proposed analysis is suitable for studying the buckling behaviour of cross-ply, simply supported, hollow cylinders subjected to combined axial compression and uniform lateral pressure and cross-ply cylindrical panels under axial compression. After successful numerical comparisons with corresponding finite element results tabulated in Noor and Peters (1989), further results are presented and discussed for cases that have not as yet been considered in the literature on the basis of fully three-dimensional considerations.

PROBLEM SPECIFICATION

Figure 1 shows the nomenclature of a circular cylindrical panel with constant thickness h and axial length L_x. The radius and the circumferential length of its middle-surface are denoted by R and L_s, respectively, so $\phi = L_S/R$ represents its shallowness angle; upon choosing $\phi = 0$ or $\phi = 2\pi$, the geometrical configuration of a flat plate or a complete circular cylinder are respectively obtained as particular cases. The axial, circumferential and normal to the middle-surface co-ordinate length parameters are denoted with x, s and z, respectively, while u, v and w represent the corresponding displacement components. It is assumed that the cylindrical panel considered is made of an arbitrary number of orthotropic linearly elastic layers whose material axes of orthotropy coincide with the axes of the adopted curvilinear co-ordinate system. It is finally assumed that, dependent on the chosen geometrical configuration and the boundary conditions imposed on the curved, as well as

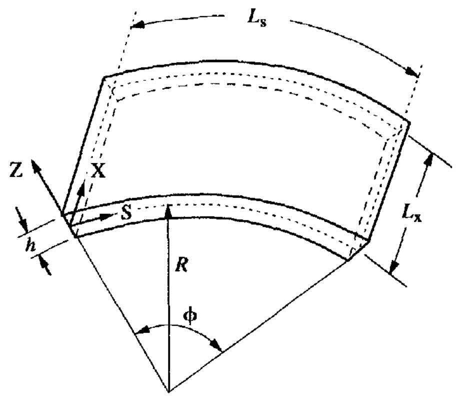

Fig. 1. Nomenclature of a circular cylindrical panel.

on the straight edges (in the case of an open panel or a flat plate), the structural element considered buckles under the single action or, where applicable, a combination of: (1) an averaged axial compression, P_x, acting at the edges $x = 0$, L_x; (2) an averaged circumferential compression, P_s, acting at the edges $s = 0$, L_S (if there are any); and (3) a suitable combination of uniform normal pressures, p^+ and p^-, acting on the outer ($z = h/2$) and inner ($z = -h/2$) lateral surfaces, respectively.

PRE-BUCKLING STATE IN A THIN ORTHOTROPIC LAYER

As it has been described in (Soldatos and Ye, 1994), with an initial displacement model of the form,

$$u = A_0 x, \quad v = B_0 s, \quad w = w_0(z), \tag{1}$$

where A_0 and B_0 are arbitrary constants, the pre-buckling state in a single orthotropic layer is assumed free of initial shear stresses. In more detail, with this displacement model two of the Navier equations of linear elasticity in cylindrical polar co-ordinates are satisfied identically. The third of these equations yields:

$$C_{33}w_0'' + C_{33}R^{-1}(1+z/R)^{-1}w_0' - C_{22}R^{-2}(1+z/R)^{-2}w_0 + R^{-1}(1+z/R)^{-1}(C_{13}-C_{12})A_0 + R^{-1}(1+z/R)^{-1}(C_{23}-C_{22})B_0 = 0, \tag{2}$$

where a prime denotes ordinary differentiation with respect to z and C_{ij} $(i, j = 1, 2, \ldots, 6)$ are material elastic constants (Jones, 1975).

The exact solution of eqn (2) has been obtained in Soldatos and Ye (1994), where the buckling problem of corresponding homogeneous orthotropic cylinders and cylindrical panels has been investigated. That exact solution can apparently be used directly in the present paper, in connection with orthotropic layers of any thickness. Dealing, however, with cross-ply laminates, it is more convenient to employ here an alternative, successive approximation approach for the exact solution of eqn (2), subject to certain boundary conditions that are introduced later [see eqns (9) and (10) below]. This approach has also been employed in connection with exact three-dimensional dynamic analyses of homogeneous and laminated composite cylinders and cylindrical panels (Soldatos and Hadjigeorgiou, 1990; Soldatos, 1991; Soldatos and Hawkes, 1991; Hawkes and Soldatos, 1992; Ye and Soldatos, 1994) and, in the next two sections, it is further employed for the solution of the associated linearized buckling equations. In the present pre-buckling stage, it yields an alternative exact solution of eqn (2) in the sense described in Soldatos (1991) where, dealing with torsional vibrations of orthotropic cylinders, it provided an alternative exact solution of a Bessel's equation.

To this end, it is assumed that the cylinder or panel considered is quite thin (i.e., $z/R \ll 1$) so that a replacement of the term $(1+z/R)$ by 1 is reasonable in eqn (2). This leads to the following approximate differential equation:

$$C_{33}w_0''+C_{33}w_0'/R-C_{22}w_0/R^2+(C_{13}-C_{12})A_0/R+(C_{23}-C_{22})B_0/R = 0, \tag{3}$$

which has constant coefficients. This can equivalently be brought into the form of the following matrix differential equation,

$$\{\mathbf{f}\}' = [\mathbf{g}]\{\mathbf{f}\}+\{\boldsymbol{\chi}\}, \quad \{\mathbf{f}\} = [\mathbf{w}_0, \boldsymbol{\sigma}_z^0]^{\mathrm{T}}, \tag{4}$$

where, the appearing initial radial stress is defined as,

$$\sigma_z^0 = C_{13}A_0+C_{23}B_0+C_{33}w_0', \tag{5}$$

and the constant elements of the 2×2 matrix $[\mathbf{g}]$ and the column vector $\{\boldsymbol{\chi}\}$ are given in the Appendix. The general solution of eqn (4) can be written in the following form,

$$\begin{aligned}\{\mathbf{f}(z)\} &= [\mathbf{b}(z)]\{\mathbf{f}(-h/2)\}+\{\boldsymbol{\psi}\}, \quad (-h/2 \leqslant z \leqslant h/2)\\ \{\boldsymbol{\psi}\} &= \int_{-h/2}^{h/2}[\mathbf{b}(z-\tau)]\{\boldsymbol{\chi}\}\,\mathrm{d}\tau,\end{aligned} \tag{6}$$

where the transfer matrix $[\mathbf{b}(z)] = \exp[(z+h/2)\mathbf{g}]$ can be evaluated analytically in various ways (Derusso *et al.*, 1965).

This pre-buckling displacement and radial stress field yield zero initial shear stresses as well as the following initial in-plane normal stresses,

$$\sigma_k^0 = \left(C_{1k}-\frac{C_{13}C_{3k}}{C_{33}}\right)A_0+\left(C_{2k}-\frac{C_{23}C_{3k}}{C_{33}}\right)\left(B_0+\frac{w_0}{R}\right)+\frac{C_{3k}}{C_{33}}\sigma_z^0, \quad (k=1,2) \tag{7}$$

where, the indices 1 and 2 in the left-hand side represent x and s, respectively. The unknown arbitrary constants A_0 and B_0 as well as the final form of the solution of eqn (4) are determined by appropriately using the stress boundary conditions imposed at the edges as well as on the two lateral surfaces of the cylinder or the cylindrical panel considered. It should be noticed again, that this will be only an approximate three-dimensional solution of the pre-buckling state considered which is satisfactorily accurate for thin layers only. As has already been mentioned, however, upon applying the successive approximation procedure described in the next sections, it eventually leads to an exact solution of eqn (2) [see eqn (19)], subject to the boundary conditions employed below. In this sense, it is equivalent to the alternative, closed form solution of eqn (2) obtained in (Soldatos and Ye, 1994).

The buckling analyses of the simply supported cylinders and cylindrical panels considered in this paper are based on two particular cases of the pre-buckling state described. Relevant buckling problems, associated with either of these cases, have not yet been considered in the literature on the basis of fully three-dimensional considerations.

(i) *Complete cylinders under combined axial compression and external pressure*

In this case, a further assumption of axisymmetric pre-buckling deformation yields $B_0 = 0$. Upon further assuming that the applied external loadings are interrelated by means of the following relation,

$$P_x = kp, \tag{8}$$

with k being a known positive constant, the boundary conditions are conveniently expressed as follows,

$$\frac{1}{h}\int_{-h/2}^{h/2} \sigma_x^0(z)\,\mathrm{d}z = -kp\,, \quad \sigma_z^0(h/2) = -p\,, \quad \sigma_z^0(-h/2) = 0. \tag{9}$$

Employing $k = 0$ or $k = \infty$ in eqn (8), eqns (9) yield boundary conditions for the pre-buckling state of a complete cylinder under a uniform external pressure or an axial compression, respectively. Although for $k \neq \infty$ the accuracy of the results is expected to be decreasing with decreasing length of the cylinder, the effects of this restriction are also expected to decrease by decreasing the value of k.

(ii) *Open cylindrical panels under axial compression*

In this case, the boundary conditions employed are as follows:

$$\frac{1}{h}\int_{-h/2}^{h/2} \sigma_x^0(z)\,\mathrm{d}z = -P_x\,, \quad \int_{h/2}^{h/2} \sigma_s^0(z)\,\mathrm{d}z = 0\,, \quad \sigma_z^0(\pm h/2) = 0. \tag{10}$$

As a particular case, three-dimensional buckling of a flat plate under axial compression can be considered by setting $\phi = 0$. However, this particular case has been studied already by Srinivas and Rao (1970) and Fan and Ye (1993).

PERTURBED STATE IN A THIN ORTHOTROPIC LAYER

Three-dimensional elasticity buckling differential equations in cylindrical polar co-ordinates are obtained by extending the variational approach described in Cartesian co-ordinates by Washizu (1975). Hence, for small displacements compared to unity and in cases where the pre-buckling state is determined by the geometrically linear analysis described in the preceding section, the elastic stability of a homogeneous orthotropic cylinder or cyl-indrical panel is described by the following linearized set of three-dimensional Navier-type equations:

$$\begin{aligned}
&C_{11}u_{,xx} + C_{66}(1+z/R)^{-2}u_{,ss} + C_{55}R^{-1}(1+z/R)^{-1}u_{,z} + C_{55}u_{,zz}\\
&\quad + (C_{12}+C_{66})(1+z/R)^{-1}v_{,xs} + (C_{13}+C_{55})w_{,xz} + (C_{12}+C_{55})R^{-1}(1+z/R)^{-1}w_{,x}\\
&\quad = \sigma_x^0 u_{,xx} + \sigma_s^0(1+z/R)^{-2}u_{,ss} + (1+z/R)^{-1}[\sigma_z^0(1+z/R)u_{,z}]_{,z},\\
&(C_{12}+C_{66})(1+z/R)^{-1}u_{,xs} + C_{66}v_{,xx} + C_{22}(1+z/R)^{-2}v_{,ss}\\
&\quad - C_{44}R^{-2}(1+z/R)^{-2}v + C_{44}R^{-1}(1+z/R)^{-1}v_{,z}\\
&\quad + C_{44}v_{,zz} + (C_{22}+C_{44})R^{-1}(1+z/R)^{-1}w_{,s}\\
&\quad + (C_{23}+C_{44})(1+z/R)^{-1}w_{,sz} = \sigma_x^0 v_{,xx} + \sigma_s^0(1+z/R)^{-2}[v_{,ss} - R^{-2}v + 2R^{-1}w_{,s}]\\
&\quad + (1+z/R)^{-1}[\sigma_z^0(1+z/R)v_{,z}]_{,z},\\
&(C_{13}+C_{55})u_{,xz} + (C_{13}-C_{12})R^{-1}u_{,x} - (C_{22}+C_{44})R^{-1}(1+z/R)^{-3}v_{,s}\\
&\quad + (C_{23}+C_{44})(1+z/R)^{-1}v_{,sz} + C_{55}w_{,xx} + C_{44}(1+z/R)^{-2}w_{,ss}\\
&\quad + C_{33}w_{,zz} - C_{22}R^{-2}(1+z/R)^{-2}w + C_{33}R^{-1}(1+z/R)^{-1}w_{,z}\\
&\quad = \sigma_x^0 w_{,xx} + \sigma_s^0(1+z/R)^{-2}[w_{,ss} - R^{-2}w + 2R^{-1}v_{,s}] + (1+z/R)^{-1}[\sigma_z^0(1+z/R)w_{,z}]_{,z},
\end{aligned} \tag{11}$$

where the appearing initial stresses are given by eqns (5) and (7). Associated to these equations, the following requirements are imposed on $z = \pm h/2$ to satisfy the boundary conditions for stress-free lateral surfaces at the perturbed buckling state,

$$\begin{aligned} C_{13}u_{,x}+C_{23}(1+z/R)^{-1}w+C_{33}w_{,z} &= 0, \\ w_{,x}+u_{,z} &= 0, \\ (1+z/R)^{-1}(w_{,s}+R^{-1}v)+v_{,z} &= 0. \end{aligned} \tag{12}$$

Due to the appearance of the terms $(1+z/R)$ as well as the initial stresses σ_x^0, σ_s^0 and σ_z^0 which are functions of z, eqns (11) are differential equations with variable coefficients. On the basis of the reasoning used in pre-buckling analysis, a replacement of $(1+z/R)$ with 1 is acceptable for quite thin cylinders ($h/R \ll 1$). Similarly, a replacement of the terms $\sigma_x^0(z), \sigma_s^0(z)$ and $\sigma_z^0(z)$ with $\sigma_x^0(R), \sigma_s^0(R)$ and $\sigma_z^0(R)$ is also reasonable. Under these considerations and with the introduction of the displacement model,

$$\begin{aligned} u &= U(z)\cos(m\pi x/L_x)\sin(n\pi s/L_s), \\ v &= V(z)\sin(m\pi x/L_x)\cos(n\pi s/L_s), \\ w &= W(z)\sin(m\pi x/L_x)\sin(n\pi s/L_s), \end{aligned} \tag{13}$$

that satisfies exactly simply supported edge boundary conditions, the above differential eigenvalue problem can be approximated as follows,

$$\{\mathbf{F}\}' = [\mathbf{G}]\{\mathbf{F}\}, \quad \{\mathbf{F}\}^{\mathrm{T}} = \{U, U', V, V', W, W'\}. \tag{14}$$

In eqn (13), n represents the circumferential full or half-wave number (in the case of a complete cylinder or an open panel, respectively), while m is the axial half-wave number of the buckling pattern. The components of the 6×6 matrix $[\mathbf{G}]$ appearing in eqn (14) are given in the Appendix and are in general dependent on the initial stress approximations $\sigma_x^0(R)$, $\sigma_s^0(R)$ and $\sigma_z^0(R)$. The general solution of eqn (14) can be explicitly expressed as,

$$\{\mathbf{F}(z)\} = [\mathbf{B}(z)]\{\mathbf{F}(-h/2)\}, \quad (-h/2 \leqslant z \leqslant h/2), \tag{15}$$

where $\{\mathbf{F}(-h/2)\}$ denotes the value of the vector $\{\mathbf{F}\}$ at the bottom surface of the cylinder ($z = -h/2$). Moreover, as has already been mentioned with the matrix $[\mathbf{b}(z)]$, for a given value of z (representing a z-surface that is parallel to the middle surface), the elements of the matrix $[\mathbf{B}(z)] = \exp[(z+h/2)\mathbf{G}]$ can be evaluated analytically (Derusso *et al.*, 1965).

On the other hand, inserting eqns (13) into the lateral boundary conditions [eqn (12)] and using the notation adopted in eqn (14) yields the following boundary conditions at $z = \pm h/2$,

$$\begin{aligned} (m\pi/L_x)C_{13}F_1+C_{23}(1+z/R)^{-1}(n\pi F_3/L_s-R^{-1}F_5)-C_{33}F_6 &= 0, \\ F_2+(m\pi/L_x)F_5 &= 0, \\ F_4(1+z/R)^{-1}(R^{-1}F_3-n\pi F_5/L_s) &= 0. \end{aligned} \tag{16}$$

These boundary conditions in connection with eqn (15) lead to an algebraic eigenvalue problem that yields an approximate solution of the buckling problem considered. However, such an approximate solution is expected to approach the exact three-dimensional solution of the problem as the thickness of the layer decreases.

SOLUTION FOR THICK OR LAMINATED CYLINDERS OR PANELS

In this respect, the solution of the governing equations [eqns (3) and (11)] is based on the division of the hollow cylinder or open panel considered into N coaxial and successive fictitious subcylinders. Different layers may have different thicknesses or material properties. However, it is assumed that the thickness of each layer approaches zero as N approaches infinity. Assuming, in addition, that each sublayer is homogeneous and made of either isotropic or orthotropic material, two types of material interfaces are distinguished in such a cross-ply laminate: the fictitious interfaces that separate layers with same material properties and the real ones that separate layers of different materials. For each of these subcylinders the approximate solutions in eqns (4) and (15) are initially formed. Upon choosing a suitably large value of N, each individual layer becomes thin and, as a result, an approximate solution of the form described in the two preceding sections is considered adequate for the study of its stability behaviour. Then, all solutions obtained are suitably connected by means of appropriate continuity conditions imposed on the fictitious and real interfaces, thus providing an arbitrary close solution of the exact governing differential equations given in eqns (3) and (11).

Dealing in particular with the interface of the jth and $(j+1)$th of the aforementioned fictitious layers, having thicknesses $h^{(j)}$ and $h^{(j+1)}$, respectively, the following continuity conditions are considered ($j = 1, 2, \ldots, N-1$),

$$w_0(-h^{(j+1)}/2) = w_0(h^{(j)}/2), \quad \sigma_z^0(-h^{(j+1)}/2) = \sigma_z^0(h^{(j)}/2), \tag{17a}$$

and,

$$\begin{aligned} U(-h^{(j+1)}/2) &= U(h^{(j)}/2), & V(-h^{(j+1)}/2) &= V(h^{(j)}/2), \\ W(-h^{(j+1)}/2) &= W(h^{(j)}/2), & \sigma_z(-h^{(j+1)}/2) &= \sigma_z(h^{(j)}/2), \\ \tau_{xz}(-h^{(j+1)}/2) &= \tau_{xz}(h^{(j)}/2), & \tau_{sz}(-h^{(j+1)}/2) &= \tau_{sz}(h^{(j)}/2), \end{aligned} \tag{17b}$$

for the pre-buckling and the buckling state, respectively. Equations (17b) are equivalent to the following continuity conditions,

$$\{\mathbf{F}^{(j+1)}(-h^{(j+1)}/2)\} = [\mathbf{T}^{(j)}]\{\mathbf{F}^{(j)}(h^{(j)}/2)\} \tag{18}$$

where the elements of the constant matrix [T] are given in the Appendix. Hence, upon recursively using eqns (6), (15), (17a) and (18), the solution for the pre-buckling and the buckling state in such an N-layered composite laminate can respectively be obtained in the following form,

$$\begin{aligned} \{\mathbf{f}^{(N)}(h^{(N)}/2)\} &= [\mathbf{b}^{(N)}(h^{(N)}/2)]\{\mathbf{f}^{(N)}(-h^{(N)}/2)\} + \{\boldsymbol{\psi}^{(N)}\} \\ &= [\mathbf{b}^{(N)}(h^{(N)}/2)]([\mathbf{b}^{(N-1)}(h^{(N-1)}/2)]\{\mathbf{f}^{(N-1)}(-h^{(N-1)}/2)\} + \{\boldsymbol{\psi}^{(N-1)}\}) + \{\boldsymbol{\psi}^{(N)}\} \\ &= \cdots = [\mathbf{A}]\{\mathbf{f}^{(1)}(-h^{(1)}/2)\} + \{\boldsymbol{\Omega}\}, \end{aligned} \tag{19}$$

$$\begin{aligned} &\{\mathbf{F}^{(N)}(h^{(N)}/2)\} = [\mathbf{B}^{(N)}(h^{(N)}/2)]\{\mathbf{F}^{(N)}(-h^{(N)}/2)\} \\ &= [\mathbf{B}^{(N)}(h^{(N)}/2)]([\mathbf{T}^{(N)}][\mathbf{B}^{(N-1)}(h^{(N-1)}/2)]\{\mathbf{F}^{(N-1)}(h^{(N-1)}/2)\}) = \cdots = [\mathbf{H}]\{\mathbf{F}^{(1)}(-h^{(1)}/2)\}, \end{aligned} \tag{20}$$

where,

$$[\mathbf{A}] = \left[\prod_{k=N}^{1} ([\mathbf{b}^{(k)}])\right], \quad [\mathbf{\Omega}] = \sum_{i=2}^{N} \left[\prod_{k=N}^{i} ([\mathbf{b}^{(k)}])\right] \{\boldsymbol{\psi}^{(i-1)}\} + \{\boldsymbol{\psi}^{(N)}\},$$

$$[\mathbf{H}] = \left[\prod_{k=N}^{1} ([\mathbf{B}^{(k)}][\mathbf{T}^{(k)}])\right]. \tag{21}$$

Equation (19), in connection with eqn (6) and the boundary conditions in eqn (9) or (10), yields a 2×2 system of linear algebraic equations from which the pre-buckling stress state is determined. Upon inserting next the obtained results into the elements of the matrix [**G**], eqn (20), in connection with the boundary conditions in eqn (16), yields an eigenvalue problem whose solution agrees with critical buckling load predictions. It is worth mentioning that, with the present formulation, the successive approximation method employed elsewhere (Soldatos and Hadjigeorgiou, 1990; Soldatos, 1991; Soldatos and Hawkes, 1991; Hawkes and Soldatos, 1992) has been converted into a recursive approach in which, independently of the number of real and/or fictitious layers involved, the eigenvalues of a 6×6 matrix are always sought.

NUMERICAL RESULTS AND DISCUSSION

Most of the numerical examples shown throughout this study are for laminated cylinders and cylindrical panels whose material properties are,

$$G_{LT}/E_T = 0.6, \quad G_{TT}/E_T = 0.5, \quad \nu_{LT} = \nu_{TT} = 0.25, \tag{22}$$

and the stiffness ratio E_L/E_T is variable. In a particular case, however, for the purpose of comparison with a previous relevant study (see Table 2), different material properties have been used.

The manner in which the present successive approximation method converges is shown in Table 1, for a two-layered, relatively thick ($h/R = 0.2$) hollow cylinder having an antisymmetric [0°/90°] lay-up. The cylinder is subjected to: (i) a pure external lateral pressure ($k = 0$); (ii) the combined action of an external pressure and an axial compression ($k = 100$); and (iii) a pure axial compression ($k = \infty$). Table 1 illustrates the influence of the number of fictitious layers, N, on the prediction of the buckling parameters,

$$\begin{aligned} p^* &= p/E_T \quad (k \neq \infty), \\ P^* &= P_x/E_T \quad (k = \infty), \end{aligned} \tag{23}$$

for a stiffness ratio $E_L/E_T = 40$. Apparently, a very fast convergence is observed in all three cases considered. Results accurate up to four significant figures are obtained for a relatively thin sub-layer's choice ($h^{(j)}/R \leqslant 0.02$). Moreover, for such relatively thick cylinders, an acceptable accuracy of buckling load predictions is achieved even without the use of successive approximations ($N = 2$). In this respect, it should be mentioned that similarly fast convergence rates have always been observed in cases that the proposed method was

Table 1. Buckling parameters p^* or P^* of a two-layered [0°/90°] hollow cylinder under different loadings ($E_L/E_T = 40$, $L_x/R = 5$, $h/R = 0.2$)

N	$k = 0$	$k = 100$	$k = \infty$
2	1.2329×10^{-2}	2.7945×10^{-3}	3.6117×10^{-1}
4	1.1947×10^{-2}	2.7600×10^{-3}	3.5872×10^{-1}
6	1.1875×10^{-2}	2.7532×10^{-3}	3.5821×10^{-1}
8	1.1849×10^{-2}	2.7508×10^{-3}	3.5804×10^{-1}
10	1.1837×10^{-2}	2.7496×10^{-3}	3.5795×10^{-1}
12	1.1831×10^{-2}	2.7490×10^{-3}	3.5790×10^{-1}

Table 2. Buckling parameter ($P_x R^2/E_T h^2$) of a 40-layered orthotropic cylinders under axial compression ($L_x/R = 5, h/R = 0.2, k = \infty$)

m	n	Noor	Present	m	n	Noor	Present
1	0	12.50	12.520	2	2	5.466	5.467
1	1	5.511	5.520	2	3	16.54	16.674
1	2	11.08	11.104	3	0	12.50	12.572
1	3	57.70	57.584	3	1	6.838	6.849
2	0	12.50	12.564	3	2	5.052	5.051
2	1	6.380	6.391	3	3	9.622	9.613

applied in connection with corresponding free vibration problems (Soldatos and Hadjigeorgiou, 1990; Soldatos, 1991; Soldatos and Hawkes, 1991; Hawkes and Soldatos, 1992; Ye and Soldatos, 1994).

For the purpose of comparison with corresponding results based on an alternative three-dimensional investigation (Noor and Peters, 1989), the buckling problem of a relatively thick ($L_x/R = 5$, $h/R = 0.2$), simply supported, 40-layered complete cylinder under axial compression is considered. The following material properties are used in the Noor and Peters (1989) finite element analysis,

$$E_L/E_T = 15, \quad G_{LT}/E_T = 0.6, \quad G_{TT}/E_T = 0.35, \quad \nu_{LT} = \nu_{TT} = 0.3, \tag{24}$$

for a cylinder having an antisymmetric $[0°/90°]_{20}$ cross-ply lay-up. For such a cylinder, Table 2 compares the dimensionless axial buckling loads ($P_x R^2/E_T h^2$) tabulated in Noor and Peters (1989) with corresponding results based on the present approach. Due to the small thickness to middle surface ratio in each layer ($h^{(i)}/R = 0.005$), further consideration of fictitious sub-layers was unnecessary. It is observed that both groups of results compared are in very close agreement. In particular, both methods provided practically identical critical buckling load predictions ($m = 3$, $n = 2$). Using, however, a very large number of degrees of freedom and therefore matrices of large dimension, the finite element analysis employed by Noor and Peters (1989) should be computationally considerably more expensive as it compares with the present approach that requires algebraic manipulations involving 6×6 matrices only.

After the aforementioned successful comparisons made, all results shown next are for cases that have not as yet been considered in the literature on the basis of fully three-dimensional considerations. Figures 2–5 deal with hollow laminated cylinders having a symmetric or an antisymmetric cross-ply lay-up and being subjected either to a simple lateral pressure ($k = 0$) or to the combined action of a lateral pressure and an axial compression ($k = 100$). In all cases, the variation of the critical pressure load parameter p^* is shown as a function of the stiffness ratio E_L/E_T as well as the number of layers involved.

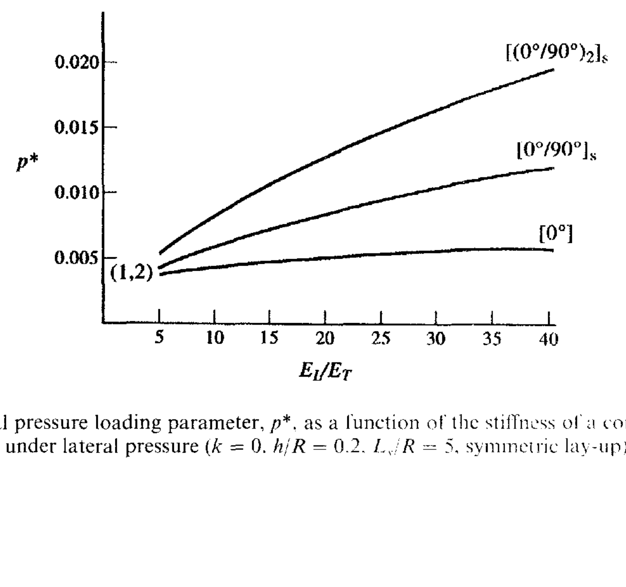

Fig. 2. Critical pressure loading parameter, p^*, as a function of the stiffness of a complete cylinder under lateral pressure ($k = 0$, $h/R = 0.2$, $L_x/R = 5$, symmetric lay-up).

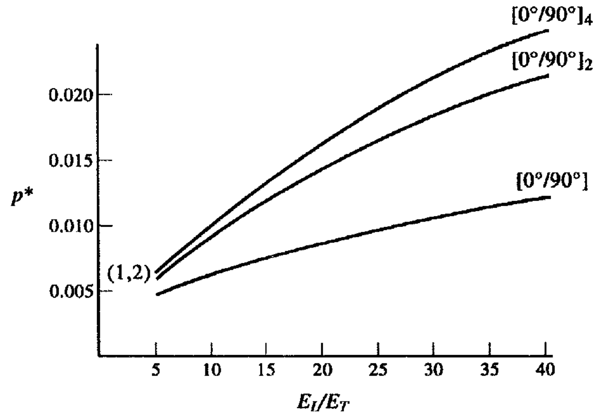

Fig. 3. Critical pressure loading parameter, p^*, as a function of the stiffness of a complete cylinder under lateral pressure ($k = 0$, $h/R = 0.2$, $L_x/R = 5$, antisymmetric lay-up).

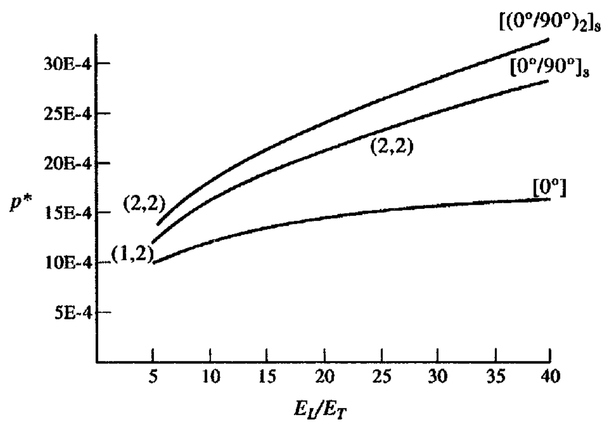

Fig. 4. Critical pressure loading parameter, p^*, as a function of the stiffness of a complete cylinder under combined lateral pressure and axial compression ($k = 100$, $h/R = 0.2$, $L_x/R = 5$, symmetric lay-up).

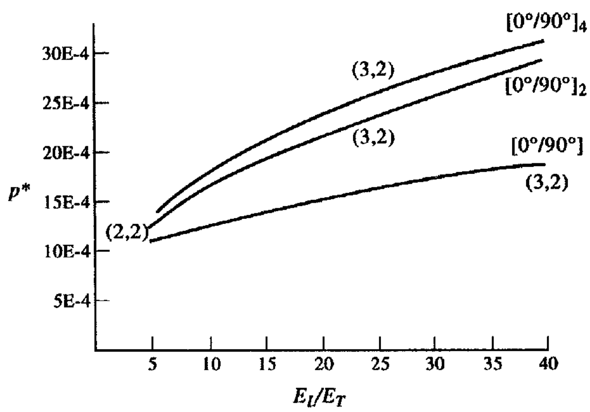

Fig. 5. Critical pressure loading parameter, p^*, as a function of the stiffness of a complete cylinder under combined lateral pressure and axial compression ($k = 100$, $h/R = 0.2$, $L_x/R = 5$, antisymmetric lay-up).

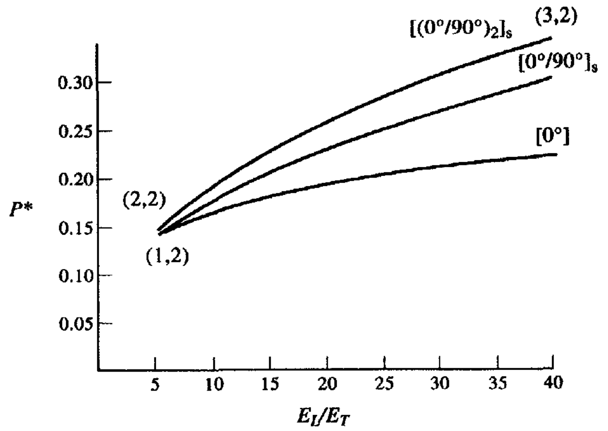

Fig. 6. Critical buckling load parameter, P^*, as a function of the stiffness of a complete cylinder under axial compression ($k = \infty$, $h/R = 0.2$, $L_x/R = 5$, symmetric lay-up).

Corresponding results dealing with the variation of the critical axial load parameter P^* ($k = \infty$) are shown in Figs 6 and 7 for cylinders subjected to a simple axial compression. The axial and circumferential wave numbers of corresponding buckling patterns are shown in these figures as (m, n). As was expected, critical buckling loads are in all cases increasing with either increasing stiffness ratio or number of layers. This observation agrees and further validates a similar trend of numerical results based on corresponding two-dimensional studies (e.g. Jones, 1975; Whitney, 1987; Soldatos, 1992b). Dealing, in particular, with antisymmetric cross-ply lay-ups (Figs 3, 5 and 7), it represents a well-known result according to which the bending–extensional coupling due to lamination dies out with increasing the number of layers.

It is further observed that: (i) cylinders under pure external lateral pressure (Figs 2 and 3) buckle always in a pattern having one axial and two circumferential half waves; (ii) upon increasing the axial compression (i.e. the value of k), the number of layers or the stiffness ratio, the axial half-wave number of the buckling pattern increases; (iii) upon increasing the axial compression (or, respectively, the external lateral pressure), the critical external pressure (or, respectively, the critical axial compression) decreases; and (iv) although for homogeneous cylinders reinforced in the axial direction (Figs 2, 4 and 6: [0°] stacking sequence), buckling occurs with one axial half-wave and one circumferential full-wave (two half-waves), upon employing circumferentially reinforced layers ($[0°/90°]_s$ and $\{[0°/90°]_2\}_s$ stacking sequences), corresponding cylinders buckle in a pattern having more than one axial half-wave.

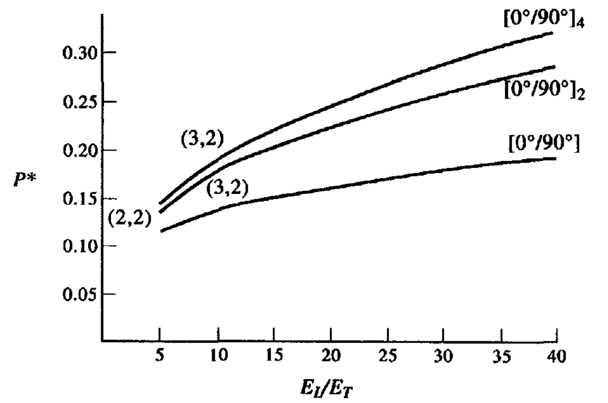

Fig. 7. Critical buckling load parameter, P^*, as a function of the stiffness of a complete cylinder under axial compression ($k = \infty$, $h/R = 0.2$, $L_x/R = 5$, antisymmetric lay-up).

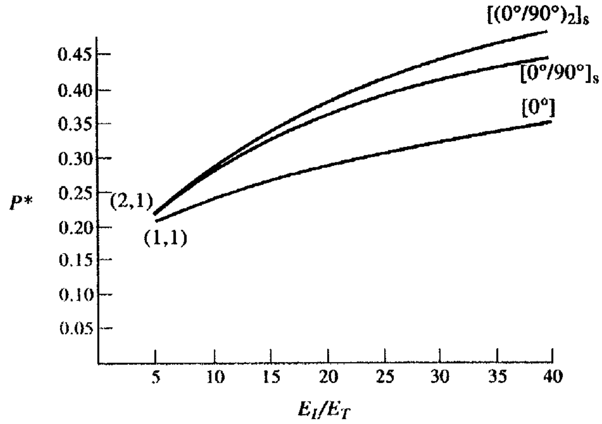

Fig. 8. Critical buckling load parameter, P^*, as a function of the stiffness of an open cylindrical panel under axial compression ($k = \infty$, $h/R = 0.2$, $L_x/R = 2$, $\phi = \pi/3$, symmetric lay-up).

In a similar manner, Figs 8 and 9 show, respectively, the variation of the critical axial compression parameter $P^*(k = \infty)$ as a function of the stiffness ratio E_L/E_T as well as the number of layers of a laminated open cylindrical panel with a symmetric and an anti-symmetric cross-ply lay-up. It is again observed that critical buckling loads increase with increasing stiffness ratio, independent of the stacking sequence or the number of layers employed. For homogeneous panels reinforced in the axial direction ([0°] stacking sequence), buckling occurs with one axial and one circumferential half-wave. Upon employing, however, circumferentially reinforced layers ($[0°/90°]_s$ and $\{[0°/90°]_2\}_s$ stacking sequences), corresponding panels buckle in a pattern having two or more axial half-waves.

CONCLUSIONS

This paper studied the buckling problem of simply supported cross-ply laminated cylinders and cylindrical panels, under certain external loadings, on the basis of fully three-dimensional considerations. In more detail, the buckling of open cylindrical panels subjected to an axial compression was investigated, while complete hollow cylinders were assumed to buckle under the combined action of an axial compression and a uniform external pressure. In all cases considered, the pre-buckling state was assumed free of initial shear

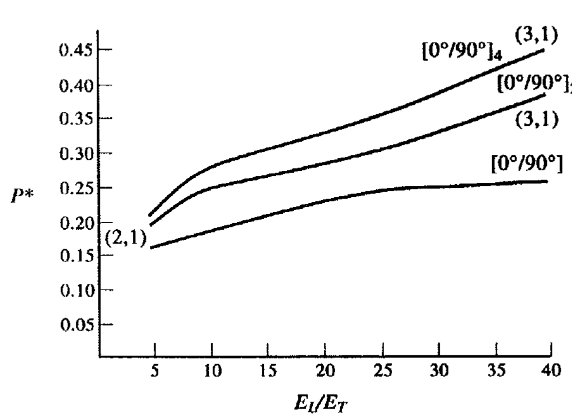

Fig. 9. Critical buckling load parameter, P^*, as a function of the stiffness of an open cylindrical panel under axial compression ($k = \infty$, $h/R = 0.2$, $L_x/R = 2$, $\phi = \pi/3$, antisymmetric lay-up).

stresses. In the case of complete cylinders, this assumption resulted in an axisymmetric pre-buckling state. Both the pre-buckling and the linearized buckling governing differential equations were solved on the basis of a recursive approach. This is equivalent to the successive approximation method employed recently in connection with exact three-dimensional dynamic analysis of homogeneous and laminated composite cylinders and panels (Soldatos and Hadjigeorgiou, 1990; Soldatos, 1991; Soldatos and Hawkes, 1991; Hawkes and Soldatos, 1992). It presents, however, a considerable advantage: buckling load predictions are always obtained as eigenvalues of a 6×6 matrix, independently of the number of successive approximations required.

In the particular case of a forty-layered complete cylinder under axial compression, buckling load predictions based on the present method were found to be in excellent agreement with corresponding results based on a three-dimensional finite element formulation (Noor and Peters, 1989). It was concluded, however, that using a very large number of degrees of freedom and, therefore, matrices of large dimension, the finite element analysis employed by Noor and Peters (1989) should be computationally considerably more expensive compared with the present approach that requires algebraic manipulations involving 6×6 matrices only.

After the successful comparisons made, further results were shown and discussed for cases that have not as yet been considered in the literature on the basis of full three-dimensional considerations. These dealt with buckling of cylinders and cylindrical panels having a symmetric or an antisymmetric cross-ply lay-up. Their trends were generally agreed and further validated similar trends of numerical results based on corresponding two-dimensional studies. However, these results are expected to be more accurate than corresponding buckling results based on such two-dimensional investigations. In this respect, the three-dimensional buckling analysis proposed in this paper can be further used for the assessment of classical and, mainly, refined shell theories as far as buckling analyses are concerned.

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APPENDIX

The elements of the matrices [g] and $\{\chi\}$ appearing in eqn (4) are given as follows,

$$\begin{aligned}
g_{11} &= -R^{-1}C_{23}/C_{33}, \quad g_{12} = 1/C_{33},\\
g_{21} &= R^{-2}(C_{22}C_{33}-C_{23}^2+C_{23}C_{33})/C_{33}, \quad g_{22} = R^{-1}(C_{23}-C_{33})/C_{33},\\
\chi_1 &= -(C_{13}A_0+C_{23}B_0)/C_{33},\\
\chi_2 &= R^{-1}[(C_{12}C_{33}-C_{13}C_{23})A_0+(C_{22}C_{33}-C_{23}^2)B_0]/C_{33}.
\end{aligned} \tag{A1}$$

The nonzero elements of the matrix [G] appearing in eqn (14) are as follows,

$$\begin{aligned}
G_{12} &= G_{34} = G_{56} = 1, \quad G_{21} = [(C_{11}-\sigma_x^0)\alpha^2+(C_{66}-\sigma_s^0)\beta^2]/(C_{55}-\sigma_z^0),\\
G_{22} &= -(C_{55}/R-\sigma_z^0/R-\mathrm{d}\sigma_z^0/\mathrm{d}z)/(C_{55}-\sigma_z^0),\\
G_{44} &= -(C_{44}/R-\sigma_z^0/R-\mathrm{d}\sigma_z^0/\mathrm{d}z)/(C_{44}-\sigma_z^0),\\
G_{66} &= -(C_{33}/R-\sigma_z^0/R-\mathrm{d}\sigma_z^0/\mathrm{d}z)/(C_{33}-\sigma_z^0), \quad G_{23} = (C_{12}+C_{66})\alpha\beta/(C_{55}-\sigma_z^0),\\
G_{25} &= -(C_{12}+C_{55})R^{-1}\alpha/(C_{55}-\sigma_z^0), \quad G_{26} = -(C_{13}+C_{55})\alpha/(C_{55}-\sigma_z^0),\\
G_{41} &= (C_{12}+C_{66})\alpha\beta/(C_{44}-\sigma_z^0),\\
G_{43} &= [(C_{66}-\sigma_x^0)\alpha^2+(C_{22}-\sigma_s^0)\beta^2+(C_{44}-\sigma_s^0)R^{-2}]/(C_{44}-\sigma_z^0),\\
G_{45} &= -(C_{22}+C_{44}-2\sigma_s^0)R^{-1}\beta/(C_{44}-\sigma_z^0), \quad G_{46} = -(C_{23}+C_{44})\beta/(C_{44}-\sigma_z^0),\\
G_{61} &= C_{12}R^{-1}\alpha/(C_{33}-\sigma_z^0), \quad G_{62} = -(C_{13}+C_{55})\alpha/(C_{33}-\sigma_z^0),\\
G_{63} &= -(C_{22}+C_{44}-2\sigma_s^0)\beta R^{-1}/(C_{33}-\sigma_z^0), \quad G_{64} = (C_{23}+C_{44})\beta/(C_{33}-\sigma_z^0),\\
G_{65} &= [(C_{55}-\sigma_x^0)\alpha^2+(C_{44}-\sigma_s^0)\beta^2+(C_{22}-\sigma_s^0)R^{-2}]/(C_{33}-\sigma_z^0),
\end{aligned} \tag{A2}$$

where $\alpha = m\pi/L_x$ and $\beta = n\pi/L_s$.

The non-zero elements of the constant matrix $[\mathbf{T}^{(j)}]$ appearing in eqn (18) are given as follows ($j = 2, 3, \ldots, N$):

$$\begin{aligned}
T_{11} &= T_{33} = T_{55} = 1, \quad T_{22} = C_{55}^{(j-1)}/C_{55}^{(j)}, \quad T_{25} = \alpha(C_{55}^{(j-1)}/C_{55}^{(j)}-1),\\
T_{43} &= (1-C_{44}^{(j-1)}/C_{44}^{(j)})\bar{R}^{-1}, \quad T_{44} = C_{44}^{(j-1)}/C_{44}^{(j)}, \quad T_{45} = \beta(C_{44}^{(j-1)}/C_{44}^{(j)}-1),\\
T_{61} &= \alpha(C_{13}^{(j)}-C_{13}^{(j-1)})/C_{33}^{(j)}, \quad T_{63} = \beta(C_{23}^{(j)}-C_{23}^{(j-1)})/C_{33}^{(j)},\\
T_{65} &= \beta(C_{23}^{(j)}-C_{23}^{(j-1)})\bar{R}^{-1}/C_{33}^{(j)}, \quad T_{66} = C_{33}^{(j-1)}/C_{33}^{(j)},
\end{aligned} \tag{A3}$$

where $\bar{R}$ is the radius of the interface between the $(j-1)$th and the jth layer and the $C_{mn}^{(j)}$ $(m, n = 1, 2, \ldots 6)$ are material constants of the jth layer $(j = 2, 3, \ldots N)$.